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The computational complexity of the elimination problem in generalized sports competitions

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Abstract

Consider a sports competition among various teams playing against each other in pairs (matches) according to a previously determined schedule. At some stage of the competition one may ask whether a particular team still has a (theoretical) chance to win the competition. The computational complexity of this question depends on the way scores are allocated according to the outcome of a match. For competitions with at most 3 different outcomes of a match the complexity is already known. In practice there are many competitions in which more than 3 outcomes are possible. We determine the complexity of the above problem for competitions with an arbitrary number of different outcomes. Our model also includes competitions that are asymmetric in the sense that away playing teams possibly receive other scores than home playing teams.

MSC: 03D15; 90C27

Keywords: Elimination problem; NP -complete; Network flow

1. Introduction

Consider a sports competition like a national soccer league in which all participating teams play against each other in pairs (matches) according to a prefixed schedule. Initially all teams have total score zero. When a team participates in a match, its total score is increased by 0 if it loses the match, by 1 if the match ends in a draw, and by 3 if it wins the match. We call $(0, 1, 3)$ the *score allocation rule* of the competition.

At a given stage of the competition one may ask whether a particular team still has a (theoretical) chance of “winning” the competition, i.e., ending up with the highest final total score. This sports competition problem (*elimination problem*) can be translated into a flow problem and would be polynomially solvable, if the ancient FIFA rule $(0, 1, 2)$ was used (cf. [9,6,3]). However, Kern and Paulusma [7] and Bernholt et al. [2] independently prove that for the rule $(0, 1, 3)$ the problem is NP -complete, and determine the computational complexity for all possible rules $(\alpha, \beta, \gamma) \in \mathbb{R}^3$ with $\alpha \leqslant \beta \leqslant \gamma$.

Other research involves Wayne [10] and Adler et al. [1], who independently present a faster algorithm for the classic elimination problem by establishing a certain elimination threshold. Gusfield and Martel [5] generalize this result for a wider range of problem settings, and study other elimination questions. In Paulusma [8] a class of the so-called competition games is introduced. In a

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Table 1
Examples of sports competitions

Set of outcomes	Competition
$\{(0,2),(1,1),(2,0)\}$	Basketball, draughts
$\{(i,25),(25,i) \mid i=0,\ldots,5\}$ $\cup\{(i,30-i) \mid i=6,\ldots,24\}$	Bridge
$\{(0,1),(\frac{1}{2},\frac{1}{2}),(1,0)\}$	Chess
$\{(0,2),(1,2),(2,1),(2,0)\}$	Darts
$\{(0,3),(1,3),(2,3),(3,2),(3,1),(3,0)\}$	Darts, volleyball
$\{(0,3),(1,1),(3,0)\}$	Draughts, soccer
$\{(0,6),(1,1),(6,0)\}$	Stratego
$\{(i,10-i) \mid i=0,\ldots,10\}$	Table-tennis
$\{(i,4-i) \mid i=0,\ldots,4\}$	Volleyball
$\{(i,5-i) \mid i=0,\ldots,5\}$	Volleyball

competition game a certain team wants to bribe some other teams in order to win the competition. The difficulty is deciding whether bribing is profitable or not, and this problem comes down to solving the related sports competition problem.

In this paper, we generalize the sports competition problem in the following ways. We allow

- Competitions with more than two different outcomes of matches,
- Competitions in which away playing teams receive other scores than home playing teams.

In both cases, we want to determine the complexity of the sports competition problem. Instead of a score allocation rule, we consider a *set of outcomes*

$$S = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$$

defining the possible outcomes of a match. For a match ending in $\alpha_i : \beta_i$, $\alpha_i \in \mathbb{R}$ is the number of points the home playing team receives, and $\beta_i \in \mathbb{R}$ is the number of points obtained by the away playing team. Note that a score allocation rule (α, β, γ) corresponds to the set of outcomes $S = \{(\alpha, \gamma), (\beta, \beta), (\gamma, \alpha)\}$. Table 1 lists several examples of competitions.

In draughts one has tried to reduce the number of draws not only by changing the number of points for a victory into 3 instead of 2 but also by making a distinction between several kinds of draws. This resulted in the following proposals, which have been tried out in several tournaments:

$$S_1 = \{(0,5),(2,3),(2\tfrac{1}{2},2\tfrac{1}{2}),(3,2),(5,0)\}$$
$$S_2 = \{(0,5),(1,3),(2,2),(3,1),(5,0)\}$$
$$S_3 = \{(0,3),(1,1),(1,2),(1,3),(2,1),(3,1),(3,0)\}.$$

In various competitions the home playing team has a certain advantage over the team of visitors (e.g., more support by the spectators, a well-known playground). Consider for example a Dutch soccer competition, in which many teams have problems to score in an away match. This motivates our second generalization, which allows to reward a victory or draw in an away match with more points. For example in case of soccer competitions an alternative set of outcomes could be $S_4 = \{(0,5),(2,3),(5,0)\}$ or $S_5 = \{(0,6),(2,3),(5,0)\}$.

We now describe the problem more precisely. Let $T = \{t_0, t_1, \ldots, t_l\}$ denote the *set of teams* participating in the competition. The particular team under consideration will be team t_0. Each team $t_i \in T$ is assigned a *current score* $s_i \in \mathbb{R}$. We refer to $s = (s_0, \ldots, s_l) \in \mathbb{R}^T$ as the *current score vector*. The *set of remaining matches* is denoted by M. A match $m \in M$ in which team t_i plays at home against t_j is denoted as $t_i : t_j$. It is possible that two matches m_1 and m_2 in M have the same home team t_i playing against the same away team t_j. The triple (T, s, M) defines an instance of the generalized sports competition problem as defined below.

Let $\tilde{s} \in \mathbb{R}^T$ denote the final score vector, i.e., $\tilde{s}_i$ is the final score for team $t_i \in T$ after all remaining matches in M have been played. We say that t_0 has *won* the competition, if $\tilde{s}_i \leqslant \tilde{s}_0$ for all $t_i \in T$. Our sports competition problem can now be formulated as

Generalized sports competition (GSC(S)).

Instance: A triple (T, s, M) as described above.

Question: Can a final score vector $\tilde{s} \in \mathbb{R}^T$ be reached such that $\tilde{s}_i \leqslant \tilde{s}_0$ for all $t_i \in T$?

Our main result completely characterizes the complexity of GSC(S) for each possible choice of S.

Remark. Note that in the definition of GSC(S) no assumption is made on the current score vector $s \in \mathbb{R}^{\mathrm{T}}$. In particular, we do not require that s can be "reached" from an initial score vector $s^* = (0, \ldots, 0)$ via previous matches with outcomes in S. Adding such a reachability condition on the current score vector—thus restricting the set of instances—would probably leave our results unchanged but complicate the proofs considerably. (For example, normalizing S as we do in Section 4 is problematic.) The problem of deciding whether a given score vector is reachable, say, from $s^* = (0, \ldots, 0)$ is probably a difficult problem in its own.

We first show that the complexity of GSC depends on the complexity of a specific subproblem of GSC, the so-called *partial sports competition problem* PSC, where t_0 has already finished all its matches, i.e., its final score equals its current score $\tilde{s}_0 = s_0$. We prove that GSC(S) is polynomially equivalent to PSC(S) and then restrict ourselves to the problem PSC.

2. The partial sports competition problem

Consider an instance (T, s, M) of GSC(S) with corresponding set of outcomes $S = \{(\alpha_i, \beta_i) \mid 1 \leqslant i \leqslant n\}$, and assume that t_0 has finished all its matches, i.e., $\tilde{s}_0 = s_0$. We can then model the partial sports competition problem as follows.

We introduce a directed multigraph $G = (V, A)$. Each vertex $i \in V$ represents a team $t_i \neq t_0$. Each vertex $i \in V$ has a *capacity*

$$c_i = s_0 - s_i (= \tilde{s}_0 - s_i)$$

indicating how many score points t_i may still get. The arcs $a = (i, k) \in A$ represent matches $t_i : t_k$. So an arc from i to k indicates that team t_i has a home match against team t_k. An *assignment* is then a map $A \to S$, assigning some outcome (α_j, β_j) to every arc $a \in A$. Thus an assignment partitions the sets $\Delta^+(i)$ and $\Delta^-(i)$ of leaving, respectively, incoming arcs at $i \in V$ into sets

$$A_i^j = \{a \in \Delta^+(i) \mid a \text{ is assigned to } \alpha_j : \beta_j\}$$

and $B_i^j = \{a \in \Delta^-(i) \mid a \text{ is assigned to } \alpha_j : \beta_j\}$.

The partial sports competition problem can then be equivalently stated as

Partial sports competition problem PSC(S).

Instance: A multigraph $G = (V, A)$ and node capacities $c \in \mathbb{R}^V$.

Question: Can we find an assignment such that for each node $i \in V$:

$$\sum_{j=1}^{n} \alpha_j |A_i^j| + \sum_{j=1}^{n} \beta_j |B_i^j| \leqslant c_i? \tag{2.1}$$

An assignment satisfying the capacity constraints (2.1) is called a *solution* of the instance (G, c).

3. Equivalence of GSC and PSC

For most sets S (e.g., for those listed in Table 1) the assumption $\tilde{s}_0 = s_0$ is easily seen to be *without loss of generality*. Indeed, when analyzing whether t_0 has still a chance of winning the competition, we may always assume w.l.o.g. that t_0 wins all its remaining matches. For arbitrary sets of outcomes, determining the "optimal" $\tilde{s}_0$ (i.e., reducing GSC(S) to PSC(S)) is somewhat more complicated.

Example 3.1. Consider, say, a competition consisting of four teams with $S = \{(1, 0), (2, 7), (0, 1)\}$. It is not immediately clear how many matches of t_0 have to end in $1 : 0$ or $0 : 1$. In general, one might think that the optimal strategy for t_0 would be to gain a rather high final score $\tilde{s}_0$ by playing "as many as possible" matches $2 : 7$. However, this is not true, if the competition is in a state such as defined below.

Teams	Scores
t_0	11
t_1	13
t_2	16
t_3	23

Remaining matches
$t_0 : t_1$
$t_1 : t_0$
$t_0 : t_2$
$t_0 : t_3$
$t_1 : t_2$

Then t_0 can only win the competition by playing 2 : 7 and 7 : 2 against t_1, 2 : 7 against t_2, and 1 : 0 against t_3 resulting in a final score $\tilde{s}_0 = 23$.

Now consider an arbitrary set $S = \{(\alpha_j, \beta_j) \mid j = 1, \ldots, n\}$ of outcomes. If PSC(S) is NP -complete, then so is (the more general) GSC(S).

So assume PSC(S) is polynomially solvable. We could then, in principle, solve GSC(S) as follows. Given an instance (T, s, M), we consider all possible ways (t_0-*assignments*) in which t_0 can finish its remaining matches. We then solve PSC(S) for the various score vectors $\bar{s}$ and corresponding capacities

$$\bar{c}_i = \bar{s}_0 - \bar{s}_i.$$

We claim that it suffices to consider only polynomially many t_0-assignments.

First, note the following: if two different t_0-assignments result in capacity vectors $\bar{c}$ and c' with $\bar{c} \leqslant c'$ (i.e., $\bar{c}$ is *dominated* by c'), then it suffices to consider the t_0-assignment leading to c'. (If PSC(S) is solvable with capacities $\bar{c}$, then it is also solvable with capacities c'.)

This observation allows us to reduce the relevant possible outcomes: suppose $(\alpha_i, \beta_i), (\alpha_j, \beta_j) \in S$ with $\alpha_i \leqslant \alpha_j$ and $\alpha_i - \beta_i \leqslant \alpha_j - \beta_j$. Then a t_0-assignment (and its corresponding capacity vector) that lets t_0 play $\alpha_i : \beta_i$ in a home match is dominated. So we may restrict the possible outcomes for home matches of t_0 to a set

$$S_0^h = \{(\alpha_{i_1}, \beta_{i_1}), \ldots, (\alpha_{i_p}, \beta_{i_p})\}$$

with $\alpha_{i_1} < \cdots < \alpha_{i_p}$ and $\alpha_{i_1} - \beta_{i_1} > \cdots > \alpha_{i_p} - \beta_{i_p}$.

Similarly, we may restrict w.l.o.g. the set of possible outcomes of an away match of t_0 to a set

$$S_0^a = \{(\alpha_{j_1}, \beta_{j_1}), \ldots, (\alpha_{j_q}, \beta_{j_q})\}$$

with $\beta_{j_1} < \cdots < \beta_{j_q}$ and $\beta_{j_1} - \alpha_{j_1} > \cdots > \beta_{j_q} - \alpha_{j_q}$.

Lemma 3.1. *Suppose $(\alpha_i, \beta_i), (\alpha_j, \beta_j) \in S_0^h$ with $\alpha_i < \alpha_j$. Then it suffices to consider t_0-assignments that assign less than $\frac{\beta_j - \beta_i}{\alpha_j - \alpha_i}$ home matches of t_0 against pairwise different teams to the outcome $\alpha_i : \beta_i$ (otherwise, the assignment is dominated).*

Proof. Consider a t_0-assignment with outcomes $\alpha_i : \beta_i$ for home matches of t_0 against a set T' of teams with $|T'| \geqslant \frac{\beta_j - \beta_i}{\alpha_j - \alpha_i}$. Let $\bar{s}$ denote the resulting score vector. For each $t \in T'$, change the outcome from$\alpha_i : \beta_i$ to $\alpha_j : \beta_j$ for exactly one of the $t_0 : t$ matches and leave all other outcomes unchanged. This results in a score vector s' with

$$s'_i = \begin{cases} \bar{s}_i + |T'|(\alpha_j - \alpha_i) & \text{if } t_i = t_0 \\ \bar{s}_i + \beta_j - \beta_i & \text{if } t_i \in T' \\ \bar{s}_i & \text{otherwise.} \end{cases}$$

Then $c' \geqslant \bar{c}$ for the corresponding capacity vector, i.e., $\bar{c}$ is dominated. □

Note that in Lemma 3.1 $\alpha_i < \alpha_j$ implies $\frac{\beta_j - \beta_i}{\alpha_j - \alpha_i} > 0$, since $\alpha_i - \beta_i > \alpha_j - \beta_j$ by construction of S_0^h. So if

$$k_0^h := \max\left\{\frac{\beta_j - \beta_i}{\alpha_j - \alpha_i} \mid (\alpha_i, \beta_i), (\alpha_j, \beta_j) \in S_0^h\right\},$$

the relevant (non-dominated) t_0-assignments for home matches of t_0 can be constructed as follows. For a fixed team $t \neq t_0$ there are, say, $|M_t| \leqslant |M|$ home matches $t_0 : t$ for t_0 and hence

$$\binom{|M_t| + p - 1}{|M_t|} \leqslant \binom{|M| + p - 1}{|M|} \leqslant p|M|^p$$

possible t_0-assignments. An outcome $\alpha_i : \beta_i$ with $\alpha_i < \alpha_{i_p}$ occurs in at most k_0^h home matches of t_0 against pairwise different teams. So there are at most $(p-1)k_0^h$ many teams $t \in T$ for which t_0 finishes a home match with $\alpha_i : \beta_i,\ \alpha_i < \alpha_{i_p}$, i.e., t_0

plays its home matches $\alpha_{i_p} : \beta_{i_p}$ against at least $|T| - (p-1)k_0^h$ many teams. So the total number of relevant t_0-assignments on home matches is bounded by

$$\sum_{i=0}^{(p-1)k_0^h} \binom{|T|}{i} p|M|^p = \mathcal{O}(|T|^{pk_0^h} p|M|^p).$$

A similar argument can be applied to the away matches of t_0. Thus, we obtain

Theorem 3.1. *For any set S of outcomes,* GSC(S) *is polynomially equivalent to* PSC(S).

4. Normalization of the set of outcomes

Our main theorem completely determines the computational complexity of PSC(S) for all possible sets of outcomes S. Before we go into that result we first prove the following proposition that reduces the number of sets of outcomes with respect to complexity questions. We let δ^+ and δ^- denote the outdegree and indegree of a node.

Proposition 4.1. *Assume* $S = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)\}$. *Then there exists a set of outcomes*

$$S' = \{(0, \beta_1'), (1, \beta_2'), (\alpha_3', \beta_3'), \ldots, (\alpha_{k-1}', \beta_{k-1}'), (\alpha_k', 0)\}$$

with $k \leqslant n,\ 1 < \alpha_3' < \alpha_4' < \cdots < \alpha_k'$ *and* $\beta_1' > \beta_2' > \cdots > \beta_{k-1}' \geqslant 1$ *such that, given an instance* (G, c) *of* PSC(S) *we can derive an equivalent instance* (G', c') *of* PSC(S'), *i.e.,* (G, c) *has a solution if and only if* (G', c') *has a solution.*

Proof. Suppose two outcomes $(\alpha_i, \beta_i), (\alpha_j, \beta_j) \in S$ exist with $\alpha_i \leqslant \alpha_j$ and $\beta_i \leqslant \beta_j$. We are searching for an assignment, in which every $i \in V$ receives a sufficiently small number of additional points. So the outcome (α_i, β_i) is always more preferable than (α_j, β_j). In other words, we may remove (α_j, β_j) from S. After deleting all redundant outcomes, we can arrange the remaining outcomes in such a way that we have obtained a set $\bar{S}$ of$k \leqslant n$ outcomes

$$\{(\bar{\alpha}_1, \bar{\beta}_1), (\bar{\alpha}_2, \bar{\beta}_2), \ldots, (\bar{\alpha}_k, \bar{\beta}_k)\}$$

with $\bar{\alpha}_1 < \bar{\alpha}_2 < \cdots < \bar{\alpha}_k$ and $\bar{\beta}_1 > \bar{\beta}_2 > \cdots > \bar{\beta}_k$.

Set $\hat{c}_i := c_i - \bar{\alpha}_1 \delta^+(i) - \bar{\beta}_k \delta^-(i)$ for each $i \in V$. Then it is clear that we have obtained an equivalent instance $(G, \hat{c})$ of PSC($\hat{S}$), where

$$\hat{S} = \{(0, \bar{\beta}_1 - \bar{\beta}_k), (\bar{\alpha}_2 - \bar{\alpha}_1, \bar{\beta}_2 - \bar{\beta}_k), \ldots, (\bar{\alpha}_k - \bar{\alpha}_1, 0)\}.$$

Assume that $\bar{\alpha}_2 - \bar{\alpha}_1 \leqslant \bar{\beta}_{k-1} - \bar{\beta}_k$. Otherwise reverse the arcs in G and the pairs $(\bar{\alpha}_i, \bar{\beta}_i)$. For all $i \in V$ divide c_i by $\bar{\alpha}_2 - \bar{\alpha}_1$. For $1 \leqslant j \leqslant k$ divide $\bar{\alpha}_j - \bar{\alpha}_1$ and $\bar{\beta}_j - \bar{\beta}_k$ by $\bar{\alpha}_2 - \bar{\alpha}_1$. This way we have obtained an equivalent instance (G', c') of PSC(S'), where

$$S' = \{(0, \beta_1'), (1, \beta_2'), (\alpha_3', \beta_3'), \ldots, (\alpha_{k-1}', \beta_{k-1}'), (\alpha_k', 0)\}$$

with $k \leqslant n,\ 1 < \alpha_3' < \alpha_4' < \cdots < \alpha_k'$ and $\beta_1' > \beta_2' > \cdots > \beta_{k-1}' \geqslant 1$. □

We call the set S' in Proposition 4.1 *normalized*. Note that a set of outcomes S can be normalized in polynomial time.

5. Three-dimensional matching graphs

In cases where PSC(S) turns out to be NP -complete we prove this by reduction from 3-dimensional matching (3DM) (cf. [4]).

3-Dimensional matching (3DM).

Instance: Three disjoint sets X, Y and W with the same number of elements q and a subset $R \subseteq X \times Y \times W$.

Question: Does there exist a *3-dimensional matching*, i.e., is there a subset of triples $R' \subseteq R$ such that R' covers each element of $X \cup Y \cup W$ exactly once?

Let (X, Y, W, R) be an instance of 3DM. The problem will be trivial, if an element $z \in X \cup Y \cup W$ does not occur in some triple $r \in R$. Therefore, we assume that all $z \in X \cup Y \cup W$ occur in some $r \in R$.

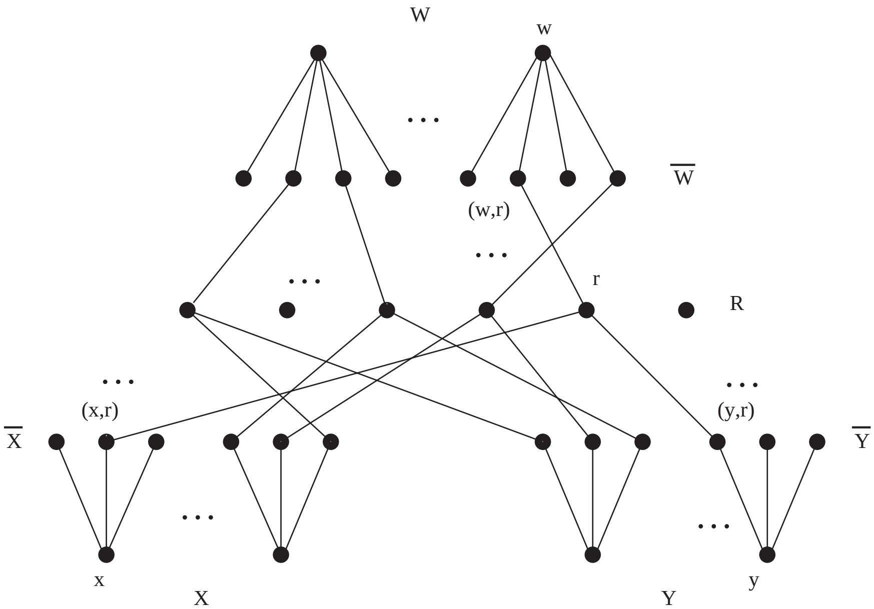

Fig. 1. A 3DM graph.

We construct an *undirected* graph $G=(V,E)$ as follows. We first make one copy of each element $z \in X \cup Y \cup W$ for each occurrence of z in R, i.e., we define

$$\begin{aligned}\bar{X} &:= \{(x,r) \mid x \in X, r \in R, x \in r\}\\ \bar{Y} &:= \{(y,r) \mid y \in Y, r \in R, y \in r\}\\ \bar{W} &:= \{(w,r) \mid w \in W, r \in R, w \in r\}.\end{aligned}$$

The node set V of G is defined as $V = X \cup Y \cup W \cup \bar{X} \cup \bar{Y} \cup \bar{W} \cup R$. The edges of G are defined by the incidence relations in a straightforward way, i.e.,

$$\begin{aligned}E = \;& \{(x,(x,r)) \mid (x,r) \in \bar{X}\}\\ & \cup \{(y,(y,r)) \mid (y,r) \in \bar{Y}\}\\ & \cup \{(w,(w,r)) \mid (w,r) \in \bar{W}\}\\ & \cup \{(r,(x,r)) \mid (x,r) \in \bar{X}\}\\ & \cup \{(r,(y,r)) \mid (y,r) \in \bar{Y}\}\\ & \cup \{(r,(w,r)) \mid (w,r) \in \bar{W}\} \quad \text{(cf. Fig. 1).}\end{aligned}$$

We call the graph G a *3-dimensional matching graph*. Our reduction from 3DM in Section 6 is based on this type of graphs by directing them and defining node capacities in an appropriate way.

6. Complexity results

The following theorem determines the complexity of PSC(S) for all sets of outcomes S.

Theorem 6.1. PSC(S) *is polynomially solvable if, after normalization,*

$$S = \{(i, n-i) \mid 0 \leqslant i \leqslant n\}$$

for some $n \in \mathbb{N}$. *In all other cases the problem is* NP *-complete.*

Proof. By Proposition 4.1 we may without loss of generality assume that S is normalized. If $|S| = 1$, then after normalization $S = \{(0,0)\}$ and the problem is trivial.

Suppose $|S|\geqslant 2$ and $S=\{(0,\beta_1),(1,\beta_2),(\alpha_3,\beta_3),\ldots,(\alpha_{n-1},\beta_{n-1}),(\alpha_n,0)\}$ with $1<\alpha_3<\alpha_4<\cdots<\alpha_n$ and $\beta_1>\beta_2>\cdots>\beta_{n-1}\geqslant 1$. We prove the theorem by establishing a sequence of claims. In the end it will be clear that only if $S=\{(i,n-i)\mid 0\leqslant i\leqslant n\}$, PSC($S$) is polynomially solvable.

Claim 1. *If $\alpha_i>\alpha_{i-1}+1$ for some $3\leqslant i\leqslant n$, then* PSC(S) *is NP-complete.*

As mentioned in the previous section we prove NP-completeness by reduction from 3DM. Suppose $|X|=|Y|=|W|=q$ and $R\subseteq X\times Y\times W$ are given. We are to determine whether R contains a matching $R'\subseteq R$. After constructing the corresponding 3DM graph G (cf. Fig. 1) we direct the edges and define node capacities $c\in\mathbb{R}^V$ as follows. δ refers to the degree function of G.

arcs from	W	to	$\bar{W}$	$c\equiv\alpha_n(\delta-1)+\alpha_{n-1}$	on W
arcs from	R	to	$\bar{W}$	$c\equiv\max\{\beta_{i-1},\beta_{n-1}+\beta_i\}$	on $\bar{W}$
arcs from	R	to	$\bar{X}$	$c\equiv\max\{\alpha_i,2+\alpha_{i-1}\}$	on R
arcs from	R	to	$\bar{Y}$	$c\equiv\beta_1+\beta_2$	on $\bar{X}\cup\bar{Y}$
arcs from	X	to	$\bar{X}$	$c\equiv 1$	on $X\cup Y$.
arcs from	Y	to	$\bar{Y}$.		

This way we have constructed an instance $(\bar{G},c)$ of PSC(S). We claim that $(\bar{G},c)$ has a solution if and only if R contains a 3DM.

"$\Leftarrow$" Suppose $R'\subseteq R$ is a matching. Define a corresponding assignment for $\bar{G}$ as follows. For each $w\in W$ choose the unique $r'\in R'$ with $(w,r')\in\bar{W}$. Let the match $w:(w,r')$ end in $\alpha_{n-1}:\beta_{n-1}$ and all other matches between w and $\bar{W}$ in $\alpha_n:0$. This way the capacity constraints of w are met. For each $r=(x,y,w)\in R'$ let $r:(w,r)$ end in $\alpha_i:\beta_i$. Both $r:(x,r)$ and $r:(y,r)$ end in $0:\beta_1$. For each $r=(x,y,w)\in R\backslash R'$ let $r:(w,r)$ end in $\alpha_{i-1}:\beta_{i-1}$. Both $r:(x,r)$ and $r:(y,r)$ end in $1:\beta_2$. This way we ensure that the capacity constraints on $\bar{W}$ and R are respected. Finally, let all matches between $\bar{X}$ and X end in $0:\beta_1$ except those that correspond to an element in R'. They end in $1:\beta_2$. This way the capacity constraints for X and $\bar{X}$ are met. We determine the outcomes of matches between $\bar{Y}$ and Y in the same way. This assignment gives a solution of the instance $(\bar{G},c)$.

"$\Rightarrow$" Conversely, suppose we are given an assignment for $\bar{G}$ respecting the capacity constraints. Each $x\in X$ has achieved at most 1 additional point. Suppose w.l.o.g. that x indeed has played one match that ended in $1:\beta_2$, while all other remaining matches between x and $\bar{X}$ ended in $0:\beta_1$. (If this is not the case, then we could modify our solution without violating the capacity constraints). A similar argument holds for elements $y\in Y$.

Nodes in $\bar{X}$ have degree 2. In view of their capacity bound $\beta_1+\beta_2$, we may assume w.l.o.g. that each $(x,r)\in\bar{X}$ has played one match that ended in $0:\beta_1$, and one match that ended in $1:\beta_2$. Otherwise we could again modify the solution, since $\beta_1>\beta_2>\cdots>\beta_{n-1}>\beta_n=0$. Then we conclude that

- There are exactly $|X|$ matches between $\bar{X}$ and R ending in $0:\beta_1$. Moreover, if $r:(x,r)$ has ended in $0:\beta_1$ and $r':(x',r')$ has ended in $0:\beta_1$, then $x\neq x'$.

The same holds for matches between $\bar{Y}$ to R.

A node $w\in W$ has capacity $\alpha_n(\delta(w)-1))+\alpha_{n-1}$. Then w.l.o.g. we may assume that $\delta(w)-1$ matches between w and $\bar{W}$ have ended in $\alpha_n:0$, and that one match of w has ended in $\alpha_{n-1}:\beta_{n-1}$. Otherwise we could modify the solution, since $\alpha_n>\alpha_{n-1}>\cdots>\alpha_3>\alpha_2=1$.

Nodes in $\bar{W}$ have degree 2 and capacity bound $\max\{\beta_{i-1},\beta_{n-1}+\beta_i\}$.

Suppose $\max\{\beta_{i-1},\beta_{n-1}+\beta_i\}=\beta_{i-1}$. If $w:(w,r)$ has ended in $\alpha_n:0$, then we may assume w.l.o.g. that $r:(w,r)$ has ended in $\alpha_{i-1}:\beta_{i-1}$. If $w:(w,r)$ has ended in $\alpha_{n-1}:\beta_{n-1}$, then the maximum number of points (w,r) could achieve in its away match against r is β_i. (Recall that $\beta_{n-1}\geqslant 1$ and $\beta_j<\beta_{j-1}$ for $2\leqslant j\leqslant n$.) Therefore, we assume that in that case $r:(w,r)$ ends in $\alpha_i:\beta_i$.

Suppose $\max\{\beta_{i-1},\beta_{n-1}+\beta_i\}=\beta_{n-1}+\beta_i$. If $w:(w,r)$ has ended in $\alpha_n:0$, then we may assume that $r:(w,r)$ does not end in $\alpha_i:\beta_i$, since we can always change the outcome into $\alpha_{i-1}:\beta_{i-1}$. If $w:(w,r)$ has ended in $\alpha_{n-1}:\beta_{n-1}$, then we can assume that $r:(w,r)$ has ended in $\alpha_i:\beta_i$.

In both cases we conclude that

- There are exactly $|W|$ matches between $\bar{W}$ and R ending in $\alpha_i:\beta_i$. Moreover, if $r:(w,r)$ has ended in $\alpha_i:\beta_i$ and $r':(w',r')$ has ended in $\alpha_i:\beta_i$, then $w\neq w'$.

Finally, the capacity constraints on R imply that a node $r=(x,y,w)\in R$ can only play a match against (w,r) that ends in $\alpha_i:\beta_i$, if both $r:(x,r)$ and $r:(y,r)$ end in $0:\beta_1$.

This can be seen as follows. If $c(r) = \alpha_i$, this is immediately clear. Suppose $c(r) = 2 + \alpha_{i-1}$. Suppose $r : (w, r)$ ended in $\alpha_i : \beta_i$ and that, say, $r : (x, r)$ ended in $1 : \beta_2$. Then $\alpha_i + 1 \leqslant 2 + \alpha_{i-1}$. Hence $\alpha_i \leqslant \alpha_{i-1} + 1$, a contradiction to our assumption $\alpha_i > \alpha_{i-1} + 1$.

From this and the above observations, it is straightforward to check that

$$R' = \{r = (x, y, w) \in R \mid r : (w, r) \text{ ended in } \alpha_i : \beta_i\}$$

actually is a 3DM.

Claim 2. *If $\beta_{i-1} > \beta_i + \beta_{n-1}$ for some $2 \leqslant i \leqslant n-1$, then* PSC($S$) *is NP -complete.*

The proof of Claim 2 is analogously to the proof of Claim 1. Reverse α_i and β_i $(i = 1, \ldots, n)$.

From now on suppose for all $2 \leqslant i \leqslant n$

$$\alpha_i \leqslant \alpha_{i-1} + 1 \quad \text{and} \quad \beta_{i-1} \leqslant \beta_i + \beta_{n-1}. \tag{6.1}$$

Claim 3. *If $\alpha_i < \alpha_{i-1} + 1$ and $\beta_{i-2} > \beta_i + \beta_{n-1}$ for some $3 \leqslant i \leqslant n$, then* PSC($S$) *is NP -complete.*

Again we prove NP -completeness by reduction from 3DM. Suppose $|X| = |Y| = |W| = q$ and $R \subseteq X \times Y \times W$ are given. After constructing the corresponding 3DM graph G we direct the edges and define node capacities $c \in \mathbb{R}^V$ as follows:

arcs from	W	to	$\bar{W}$	$c \equiv 1$	on W
arcs from	R	to	$\bar{W}$	$c \equiv \beta_1 + \beta_2$	on $\bar{W}$
arcs from	R	to	$\bar{X}$	$c \equiv \max\{2\alpha_i, 2\alpha_{i-1} + 1\}$	on R
arcs from	R	to	$\bar{Y}$	$c \equiv \beta_i + \beta_{n-1}$	on $\bar{X} \cup \bar{Y}$
arcs from	X	to	$\bar{X}$	$c \equiv \alpha_n(\delta - 1) + \alpha_{n-1}$	on $X \cup Y$.
arcs from	Y	to	$\bar{Y}$.		

This way we have constructed an instance $(\bar{G}, c)$ of PSC(S). We claim that $(\bar{G}, c)$ has a solution if and only if R contains a 3DM.

"⇐" Suppose $R' \subseteq R$ is a matching. Define a corresponding assignment for $\bar{G}$ in a similar way as in the proof of Claim 1.

"⇒" Suppose we are given an assignment for $\bar{G}$ respecting the capacity constraints. Each $w \in W$ can achieve at most 1 additional point. Suppose w.l.o.g. that w indeed has played one match that ended in $1 : \beta_2$, while all other remaining matches between w and $\bar{W}$ ended in $0 : \beta_1$.

Nodes in $\bar{W}$ have degree 2. In view of their capacity bound $\beta_1 + \beta_2$, we may assume w.l.o.g. that each $(w, r) \in \bar{W}$ has played one match that ended in $0 : \beta_1$, and one match that ended in $1 : \beta_2$. Then we conclude that

- There are exactly $|W|$ matches between $\bar{W}$ and R ending in $0 : \beta_1$. Moreover, if $r : (w, r)$ has ended in $0 : \beta_1$ and $r' : (w', r')$ has ended in $0 : \beta_1$, then $w \neq w'$.

A node $x \in X$ has capacity $\alpha_n(\delta(x) - 1) + \alpha_{n-1}$. Then w.l.o.g. we may assume that $\delta(x) - 1$ matches between x and $\bar{X}$ have ended in $\alpha_n : 0$, and that one remaining match of x has ended in $\alpha_{n-1} : \beta_{n-1}$. A similar argument holds for elements $y \in Y$.

Nodes in $\bar{X}$ have degree 2 and capacity bound $\beta_i + \beta_{n-1}$. If $x : (x, r)$ has ended in $\alpha_n : 0$, then the maximum number of points (x, r) could achieve in its away match against r is β_{i-1}. (Recall that $\beta_{i-2} > \beta_i + \beta_{n-1}$.) By (6.1) we may assume that $r : (x, r)$ ends in $\alpha_{i-1} : \beta_{i-1}$. If $x : (x, r)$ has ended in $\alpha_{n-1} : \beta_{n-1}$, then we can assume that $r : (x, r)$ has ended in $\alpha_i : \beta_i$. Hence we conclude the following.

- There are exactly $|X|$ matches between $\bar{X}$ and R ending in $\alpha_i : \beta_i$. Moreover, if $r : (x, r)$ has ended in $\alpha_i : \beta_i$ and $r' : (x', r')$ has ended in $\alpha_i : \beta_i$, then $x \neq x'$.

The same holds for matches between $\bar{Y}$ to R.

Finally, the capacity constraints on R imply that a node $r = (x, y, w) \in R$ can only play a match against (x, r) or against (y, r) that ends in $\alpha_i : \beta_i$, if $r : (w, r)$ ends in $0 : \beta_1$.

This can be seen as follows. Suppose $r : (w, r)$ ends in $1 : \beta_2$ and $r : (x, r)$ ends in $\alpha_i : \beta_i$. The match $r : (y, r)$ ends in $\alpha_i : \beta_i$ or $\alpha_{i-1} : \beta_{i-1}$. If $c(r) = 2\alpha_i$ and $r : (y, r)$ ends in $\alpha_{i-1} : \beta_{i-1}$, then $\alpha_i + \alpha_{i-1} + 1 \leqslant 2\alpha_i$, a contradiction to our assumption $\alpha_i < \alpha_{i-1} + 1$. If $c(r) = 2\alpha_{i-1} + 1$ and $r : (y, r)$ ends in $\alpha_{i-1} : \beta_{i-1}$, then again $c(r)$ is too small.

From this and the above observations, it is straightforward to check that

$$R' = \{r = (x, y, w) \in R \mid r : (w, r) \text{ ended in } 0 : \beta_1\}$$

is a 3DM.

Claim 4. *If $\beta_{i-1} < \beta_i + \beta_{n-1}$ and $\alpha_{i+1} > \alpha_{i-1} + 1$ for some $2 \leqslant i \leqslant n-1$, then* PSC($S$) *is NP -complete.*

The proof of Claim 4 is analogously to the proof of Claim 3. Reverse α_i and β_i $(i = 1, \ldots, n)$.

Claim 5. *If $\alpha_i < \alpha_{i-1} + 1$ for some $3 \leqslant i \leqslant n$, then* PSC($S$) *is NP -complete.*

Suppose $\alpha_k < \alpha_{k-1} + 1$ with $k = \min\{i \mid \alpha_i < \alpha_{i-1} + 1\}$. Since $\alpha_2 = \alpha_1 + 1$, $k \geqslant 3$. If $\beta_{k-2} > \beta_k + \beta_{n-1}$, then the claim follows from Claim 3. Suppose $\beta_{k-2} \leqslant \beta_k + \beta_{n-1}$. Since $\beta_k < \beta_{k-1}$, we obtain $\beta_{k-2} < \beta_{k-1} + \beta_{n-1}$. Because k is minimal, $\alpha_{k-1} = \alpha_{k-2} + 1$. Hence $\alpha_k > \alpha_{k-1} = \alpha_{k-2} + 1$, and the claim follows from Claim 4.

Claim 6. *If $\beta_{j-1} < \beta_j + \beta_{n-1}$ for some $2 \leqslant j \leqslant n-1$, then* PSC($S$) *is NP -complete.*

Suppose $\beta_{j-1} < \beta_j + \beta_{n-1}$ for some $2 \leqslant j \leqslant n-1$. By (6.1) and Claim 5 we can assume that $\alpha_i = \alpha_{i-1} + 1$ for all $2 \leqslant i \leqslant n$. Then $\alpha_{j+1} = \alpha_j + 1 > \alpha_{j-1} + 1$, and the claim follows from Claim 4.

Up to now we have proven that PSC(S) is NP -complete unless

$$S = \{(i, (n-i)\beta) \mid 0 \leqslant i \leqslant n\}$$

for some $\beta \geqslant 1$.

Claim 7. *If $\beta > 1$, then* PSC(S) *is NP -complete.*

Again, we prove NP -completeness by reduction from 3DM. Suppose $|X| = |Y| = |W| = q$ and $R \subseteq X \times Y \times W$ are given. After constructing the corresponding 3DM graph G we direct the edges and define node capacities $c \in \mathbb{R}^V$ as follows:

arcs from	$\bar{W}$	to	W	$c \equiv n\beta(\delta - 1) + (n-1)\beta$	on W
arcs from	$\bar{W}$	to	R	$c \equiv n$	on $\bar{W}$
arcs from	R	to	$\bar{X}$	$c \equiv \max\{\beta, 2\}$	on R
arcs from	R	to	$\bar{Y}$	$c \equiv n\beta + (n-1)\beta$	on $\bar{X} \cup \bar{Y}$
arcs from	X	to	$\bar{X}$	$c \equiv 1$	on $X \cup Y$.
arcs from	Y	to	$\bar{Y}$.		

This way we have constructed an instance $(\bar{G}, c)$ of PSC(S). The claim that $(\bar{G}, c)$ has a solution if and only if R contains a matching can be proven in the same way as we did for Claims 1 and 3.

From the above we conclude that PSC(S) is NP -complete, if $S \neq \{(i, n-i) \mid 0 \leqslant i \leqslant n\}$ after normalization. We have proven our theorem by showing the validity of the final claim.

Claim 8. *If $\beta = 1$, then* PSC(S) *is polynomially solvable.*

Consider an instance given by $G = (V, E)$ and $c \in \mathbb{R}^V$. Construct a directed bipartite graph with node sets V and E and arcs linking each $i \in V$ to all edges in E incident with i in G. Then add an additional source s and sink t as indicated in Fig. 2.

The arcs from s to V all get lower capacity 0 and upper capacity $\lfloor c_i \rfloor$ $(i \in V)$. The arcs from V to E get lower capacity 0 and upper capacity n. The arcs from E to t get lower and upper capacity n. The resulting network has a feasible s-t flow if and only if our instance (G, c) has a solution. This can be seen as follows. Since all capacities are integral, a feasible s-t flow may also be assumed to be integral. Each node $e \in E$ in our network has two incoming arcs which carry a total flow of n units, distributed as $i : n - i$ for some $0 \leqslant i \leqslant n$ corresponding to an outcome $i : n - i$.

As a result of this and Theorem 3.1 we find the following corollary that generalizes the complexity results in [7,2].

Corollary 6.1. GSC(S) *is polynomially solvable if, after normalization,*

$$S = \{(i, n-i) \mid 0 \leqslant i \leqslant n\}$$

for some $n \in \mathbb{N}$. In all other cases the problem is NP *-complete.*

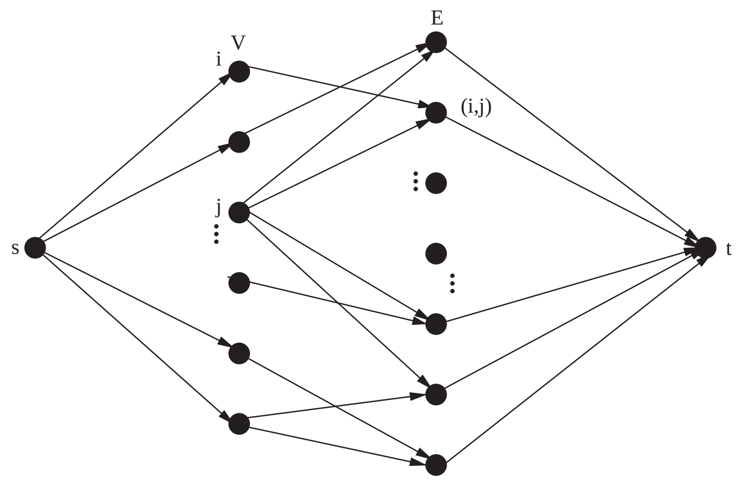

Fig. 2. The case where PSC(S) can be translated into a flow problem.

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